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Buckling of Laminated Cylindrical Shells in Terms of Different Theories and Formulations

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Bifurcation buckling analysis of laminated cylindrical shells is presented on the basis of three different shell theories: Donnell's (Donnell, L. H., "Stability of Thin-Walled Tubes under Torsion," NACA TR-479, 1933), Sanders's (Sanders, J. L., Jr., "Nonlinear Theories of Thin Shells," *Quarterly Journal on Applied Mathematics*, Vol. 21, No. 1, 1963, pp. 21–36), and Timoshenko's (Timoshenko, S., *Theory of Elastic Stability*, McGraw–Hill, New York, 1961). Formulations in terms of the displacement components and of the Airy stress function and normal displacement are examined. The partial differential equations are derived via the variational principle and solved with variables expanded in Fourier series in the circumferential direction and presented as finite differences in the axial direction. The buckling behavior of an angle-ply laminated cylindrical shell under different modes of loading was investigated parametrically, showing—in contrast to its isotropic counterparts—a discrepancy between the two formulations.

Introduction

COMPOSITE laminated cylindrical shells are widely used in industrial applications. Their buckling and postbuckling behavior is a vital safety consideration, and improvement of its prediction accuracy behavior is thus essential for reliable design.

The nonlinear behavior of shell-like structures is generally characterized by a limit point rather than by a bifurcation point (classical eigenproblems), and a nonlinear equilibrium state of the path is naturally involved. However, bifurcation analysis is still useful as a guideline and as a basic procedure for examining the imperfection sensitivity by which the limit point is characterized.[1,2] Also, the analytical buckling loads can serve as the design loads, using the suitable reduction factors.

Extensive research on the buckling and postbuckling behavior of laminated cylindrical shells is reported in literature (see the review papers of Tennyson[3] and Simitses[4]). Most of them use the simplest shell theory based on the Donnell[5] approximation. Because of the approximate nature and extreme simplicity of Donnell's equations, doubts have been raised as to their accuracy. Hoff[6] gave the range of certain basic parameters of isotropic cylindrical shells for which the solution to Donnell's and Flugge's[7] equations are approximately the same. Additional comparisons of shell theories have been published (for details, see Refs. 8 and 9); almost all of them deal with isotropic cylindrical shells or with a specific orthotropic configuration. Simitses et al.[8,9] compared the results for angle-ply laminated cylindrical shells under axial compression according to Donnell's and Sanders's shell theories, with the buckling defined as the onset of nonlinear behavior yielding the limit point depending on the imperfection shapes. No comparison is available for classical buckling of arbitrary laminated cylindrical shells under any kind of loading nor for more accurate theories than that of Sanders.

The kinematic relations yield three relatively simple equilibrium equations in the axial circumferential and normal directions using the stiffness method; these equations are written in terms of the three displacement components (axial u, circumferential v, and normal w). The main advantage of the Donnell approach is the possibility of recourse to the Airy stress function F, whereby the number of unknown functions is reduced from three (u, v, w) to two (w, F). This, however, is offset by the well-known drawback that the essential boundary conditions cannot be completely satisfied [the displacement (u, v) conditions being replaced by their derivatives]. Another problem entailed by recourse to the stress function is that reduction of the order from three to two can be reflected in the characteristic behavior, namely, whether the most important parameter—the lowest eigenvalue—is indeed encompassed. This problem, to date not treated in the literature, is especially typical for the buckling and frequency domain of laminated cylindrical shells.

The present paper has primarily two objectives: first, comparison of the bifurcation buckling behavior of an arbitrary laminated cylindrical shell under different modes of loading, using Donnell's, Sanders's, and Timoshenko's theories and thereby assessing their relative accuracies; second, investigation of the validity of the w-F formulation vs its u-v-w counterpart.

The equations for the three theories are derived on the basis of their kinematic approach for a laminated cylindrical shell under arbitrary loading, and the u-v-w and w-F formulations are developed for the Donnell-type equations.

The solution procedure is based on expansion of the variables in Fourier series in the circumferential direction and their presentation as finite differences in the axial direction. The Galerkin procedure is used to minimize the errors caused by the truncated series. Parametric analysis for an angle-ply laminated cylindrical shell validates the shell theory, but comparison of the two formulations shows that the w-F fails to bring out the lowest buckling load.

Governing Equations

The governing equations for a composite laminated cylindrical shell are derived for the Donnell,[5] Sanders,[10] and Timoshenko[11] kinematic relations. They are obtained via the variational principle for axial compression, torsion, and hydrostatic pressure. Formulation of the three approaches is based on the displacement components in the axial u, circumferential v, and normal w directions–(hereinafter referred to as the u-v-w formulation).

For the Donnell-type approach the well-known w-F formulation, based on recourse to the Airy stress function as an unknown, is considered with a view to comparison these two formulations.

Kinematics

Let (x, θ) be the cylindrical coordinates of the reference surface (x along the axial direction and θ the circumferential angle) and z the outward normal coordinate. Recourse to the Kirchhoff–Love hypothesis leaves only three dependent variables, namely, the displacement u, v, and w in the x, θ, and z directions, respectively. Under the Donnell,[5] Sanders,[10] and Timoshenko[11] kinematic approaches the strain displacement can be written as

$$\{\varepsilon\} = \{\bar{\varepsilon}\} + z\{\chi\} \tag{1}$$

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where $\{\bar{\varepsilon}\}$ and $\{\chi\}$ are, respectively, the strain of the reference surface and change-of-curvature vectors:

$$\{\bar{\varepsilon}\} = \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{\theta\theta} \\ \bar{\gamma}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} u_{,x} + \frac{1}{2}w_{,x}^2 + \delta_2\frac{v_{,x}^2}{2} \\ \frac{v_{,\theta}}{R} + \frac{w}{R} + \frac{w_{,\theta}^2}{2R^2} + \delta_1\left(\frac{v^2}{2R^2} - \frac{vw_{,\theta}}{R^2}\right) + \delta_2\frac{v_{,\theta}^2}{2R^2} \\ \frac{u_{,\theta}}{R} + v_{,x} + \frac{w_{,x}w_{,\theta}}{R} - \delta_1\left(\frac{vw_{,x}}{R}\right) + \delta_2\frac{v_{,x}v_{,\theta}}{R} \end{Bmatrix}$$

$$\{\chi\} = \begin{Bmatrix} \chi_{xx} \\ \chi_{\theta\theta} \\ 2\chi_{x\theta} \end{Bmatrix} = \begin{Bmatrix} -w_{,xx} \\ -\frac{w_{,\theta\theta}}{R^2} + \delta_1\frac{v_{,\theta}}{R^2} \\ -2\frac{w_{,x\theta}}{R} + \delta_1\frac{v_{,x}}{R} \end{Bmatrix} \tag{2}$$

$(\,)_{,x}$ and $(\,)_{,\theta}$ denote the derivatives with respect to the axial and circumferential coordinate, respectively; R is the radius of the cylinder; and δ_1 and δ_2 are introduced for the purpose of investigating the various shell theories: $\delta_1 = \delta_2 = 0$ for the Donnell kinematic relations; $\delta_1 = 1$, $\delta_2 = 0$ for the Sanders kinematic relations; $\delta_1 = 1$, $\delta_2 = 1$ for the Timoshenko kinematic relations.

The Timoshenko approach contains some additional terms, but the most dominant one is caused by $\delta_2 = 1$, especially under axial compression. Furthermore, the terms of δ_2 in $\bar{\varepsilon}_{\theta\theta}$ and in $\bar{\gamma}_{x\theta}$ are sometimes neglected, depending on the external loading type.

Constitutive Equations

Under the classical laminate theory, the constitutive equation reads

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon} \\ \chi \end{Bmatrix} \tag{3}$$

$\{N\} = \{N_{xx}, N_{\theta\theta}, N_{x\theta}\}$ and $\{M\} = \{M_{xx}, M_{\theta\theta}, M_{x\theta}\}$ being the membrane force and bending moment vectors. The coefficients of the elastic matrix are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_z Q_{ij}(1, z, z^2)\,\mathrm{d}z \tag{4}$$

A_{ij}, B_{ij}, and D_{ij} being, respectively, the membrane, coupling, and flexural rigidities, and Q_{ij} the laminate transformed reduced stiffness.

For the formulation of w-F, the relevant elastic matrices are

$$a = A^{-1}, \qquad b = A^{-1}B, \qquad d = D - BA^{-1}B \tag{5}$$

Equilibrium Equations

The equilibrium equations and the appropriate boundary conditions are derived by applying the variational principle:

$$\delta\pi = \int_{x,\theta} \{N_{xx}\delta\bar{\varepsilon}_{xx} + N_{\theta\theta}\delta\bar{\varepsilon}_{\theta\theta} + N_{x\theta}\delta\bar{\gamma}_{x\theta} + M_{xx}\delta\chi_{xx} + M_{\theta\theta}\delta\chi_{\theta\theta} + 2M_{x\theta}\delta\chi_{x\theta} - q_{xx}\delta u - q_{\theta\theta}\delta v - q_{zz}\delta w\} = 0 \tag{6}$$

where q_{xx}, $q_{\theta\theta}$, and q_{zz} are the external distributed loading in the axial, circumferential, and normal directions, respectively. With Eq. (2) substituted in Eq. (6), application of Gauss's theorem yields the following equilibrium equation:

$$N_{xx,x} + \frac{N_{x\theta,\theta}}{R} + q_{xx} = 0 \tag{7a}$$

$$N_{x\theta,x} + \frac{N_{\theta\theta,\theta}}{R} + \delta_1\left[\frac{M_{\theta\theta,\theta}}{R^2} + \frac{M_{x\theta,x}}{R} + \frac{N_{\theta\theta}}{R^2}(w_{,\theta} - v) + \frac{N_{x\theta}}{R}w_{,x}\right] + \delta_2\left[(N_{xx}V_{,x})_{,x} + \frac{(N_{\theta\theta}V_{,\theta})_{,\theta}}{R^2} + \frac{(N_{x\theta}V_{,x})_{,\theta}}{R} + \frac{(N_{x\theta}V_{,\theta})_{,x}}{R}\right] + q_{\theta\theta} = 0 \tag{7b}$$

$$M_{xx,xx} + \frac{2M_{x\theta,x\theta}}{R} + \frac{M_{\theta\theta,\theta\theta}}{R^2} - \frac{N_{\theta\theta}}{R} + (N_{xx}w_{,x})_{,x} + \frac{(N_{\theta\theta}w_{,\theta})_{,\theta}}{R^2} + \frac{(N_{x\theta}w_{,x})_{,\theta}}{R} + \frac{(N_{x\theta}w_{,\theta})_{,x}}{R} - \delta_1\left[\frac{(N_{\theta\theta}v)_{,\theta}}{R^2} + \frac{(N_{x\theta}v)_{,x}}{R}\right] + q_{zz} = 0 \tag{7c}$$

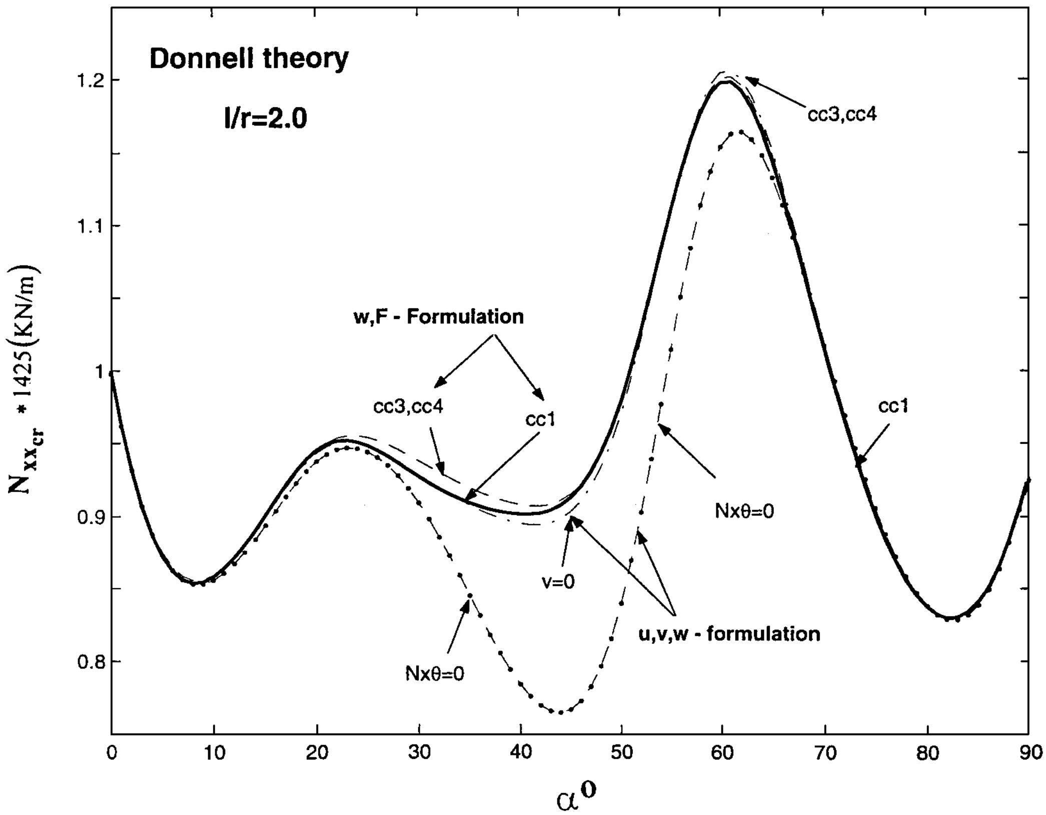

Fig. 1 Buckling load vs angle-ply ($\pm\alpha$) for $\ell/R = 2$ with clamped boundary conditions.

with the following boundary conditions:

$$u \quad \text{or} \quad N_{xx} \tag{8a}$$

$$v \quad \text{or} \quad N_{x\theta} + \delta_1(M_{x\theta}/R) + \delta_2(N_{xx}v_{,x} + N_{x\theta}v_{,\theta}/R) \tag{8b}$$

$$w \quad \text{or} \quad M_{xx,x} + 2M_{x\theta,\theta}/R + N_{xx}w_{,x} + N_{x\theta}w_{,\theta}/R - \delta_1 N_{x\theta}v/R \tag{8c}$$

$$w_{,x} \quad \text{or} \quad M_{xx} \tag{8d}$$

It should be noted here that the Timoshenko approach omits some terms of δ_2 from Eqs. (7b) and (8b).

For the Donnell-type equations, namely, $\delta_1 = \delta_2 = 0$ with $q_{xx} = q_{\theta\theta} = q_{zz} = 0$, the number of unknown functions is commonly reduced by recourse to the Airy stress functions F, but the applicability of this procedure for composite laminated cylindrical shell in the buckling domain is questionable. For the purpose of comparison in the context of the buckling behavior, two formulations of the equilibrium equations are considered: the u-v-w and w-F.

u-v-w Formulations

From the kinematic relations [Eqs. (2)] and the constitutive equations [Eqs. (3)], the equilibrium equations (7) are obtainable in terms of the displacement functions $u(x,\theta)$, $v(x,\theta)$, and $w(x,\theta)$, using the following differential operators:

$$\begin{aligned}
&L^{(e)}(u) + L^{(e)}(v) + L^{(e)}(w) + LL^{(e)}(w,w) + \eta_3\big[LL^{(e)}(w,u) \\
&\quad + LL^{(e)}(w,v) + LLL^{(e)}(w,w,w)\big] + \delta_1\big\{LL^{(e)}(v,v) \\
&\quad + (1-\eta_3)LL^{(e)}(w,v) + (1-\eta_1)LL^{(e)}(u,v) \\
&\quad + \eta_2 LL^{(e)}(w,u) + (1-\eta_1)\big[LLL^{(e)}(w,v,w) \\
&\quad + LLL^{(e)}(v,w,v) + LLL^{(e)}(v,v,v) \\
&\quad + \eta_2 LLL^{(e)}(w,w,w)\big]\big\} + q^{(e)} = 0, \qquad e = 1,2,3
\end{aligned} \tag{9}$$

The superscript e denotes the equation number [$e = 1$ for equilibrium in the x direction, Eq. (7a); $e = 2$ in the θ direction, Eq. (7b); and $e = 3$ in the normal direction, Eq. (7c)], $q^{(1)} = q_{xx}$, $q^{(2)} = q_{\theta\theta}$, and $q^{(3)} = q_{zz}$. $\eta_r = 1$ for $e = r$; otherwise zero (for example, $\eta_3 = 1$ for the third equation and $\eta_3 = 0$ for the first and second). $L^{(e)}$, $LL^{(e)}$, and $LLL^{(e)}$ are linear, quadratic, and cubic differential operators given by

$$L^{(e)}(S) = \sum_{i=0}^{4}\sum_{j=0}^{4-i} p_{ij}^{(e)} \frac{\partial^{(i+j)}S}{\partial x^{(i)}\partial\theta^{(j)}}$$

$$LL^{(e)}(S,T) = \sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{\ell=0}^{2}\sum_{k=0}^{2-\ell} p_{ijk\ell}^{(e)} \frac{\partial^{(i+j)}s}{\partial x^{(i)}\partial\theta^{(j)}} \frac{\partial^{(\ell+k)}T}{\partial x^{(\ell)}\partial\theta^{(k)}}$$

$$\begin{aligned}
LLL^{(e)}(S,T,S) = &\sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{\ell=0}^{2}\sum_{k=0}^{2-\ell}\sum_{m=0}^{1}\sum_{n=0}^{1-m} p_{ijk\ell mn}^{(e)} \frac{\partial^{(i+j)}S}{\partial x^{(i)}\partial\theta^{(j)}} \\
&\times \frac{\partial^{(\ell+k)}T}{\partial x^{(\ell)}\partial\theta^{(k)}} \frac{\partial^{(m+n)}S}{\partial x^{(m)}\partial\theta^{(n)}}
\end{aligned} \tag{10}$$

where $p_{ij}^{(e)}$, $p_{ijk\ell}^{(e)}$, and $p_{ijk\ell mn}^{(e)}$ are the coefficients of the elastic parameters, functions of A_{ij}, B_{ij}, D_{ij} of the radius R and of δ_1 and δ_2. Most of them are zero; regarding to the nonzero values some regularities can be observed, such as $p_{ij}^{(e)} \neq 0$ for the first and second equations ($e = 1$ or 2) only for the combination of $i + j = 2$ operating on u or v and $i + j = 1$ or $= 3$ on w. In a similar manner $p_{ij}^{(3)} \neq 0$ only for $i + j = 1$ or 3 operating on u or v and $i + j = 0$, 2, or 4 on w. Finally, the symbol of zero derivatives means the value itself (i.e., $\partial^{(0)}S/\partial x^{(0)}\partial\theta^{(0)} = S$, $\partial^1 S/\partial x^0\partial\theta = \partial S/\partial\theta$ and so on). The boundary conditions [Eqs. (8)] are also written in a similar way in terms of the u, v, and w functions.

w-F Formulation

The w-F formulation can be found in Refs. 12–14, but for the sake of completeness it will be recapitulated here in brief.

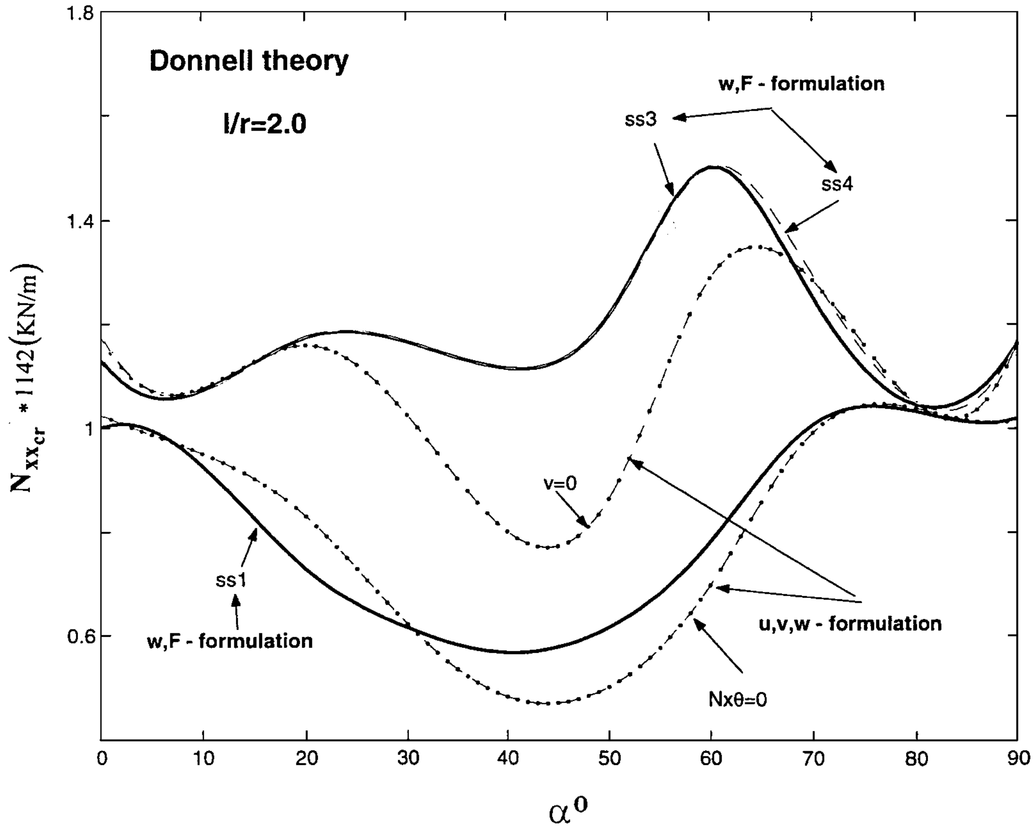

Fig. 2 Buckling load vs angle-ply (± α) for $\ell/R = 2$ with simply supported boundary conditions.

Introducing the Airy stress function $F(x, \theta)$, defined by

$$N_{xx}(x,\theta) = [F_{,\theta\theta}(x,\theta)]\big/R^2 - \bar{N}_{xx}$$

$$N_{\theta\theta}(x,\theta) = F_{,xx}(x,\theta) - \bar{N}_{\theta\theta}$$

$$N_{x\theta}(x,\theta) = -[F_{,x\theta}(x,\theta)]/R + \bar{N}_{x\theta} \tag{11}$$

The in-plane Donnell type of equilibrium equations (7a) and (7b) are identically satisfied. $\bar{N}_{xx}$, $\bar{N}_{\theta\theta}$, and $\bar{N}_{x\theta}$ represent the axisymmetric prebuckling state. The relevant equations are those of compatibility

$$L_q(w) + L_g(F) + \tfrac{1}{2}LL(w,w) - w_{,xx}/R = 0 \tag{12a}$$

and of equilibrium in the normal direction

$$L_h(w) + L_q(F) + LL(w,F) - (F_{,xx} + \bar{N}_{\theta\theta})/R + q_{zz} = 0 \tag{12b}$$

where L_p and LL are differential operators defined as

$$L_p(S) = p_{40}S_{,xxxx} + p_{31}S_{,xxx\theta} + p_{22}S_{,xx\theta\theta} + p_{13}S_{,x\theta\theta\theta} + p_{04}S_{,\theta\theta\theta\theta}$$

$$LL(S,T) = S_{,xx}T_{,\theta\theta} - 2S_{,x\theta}T_{,x\theta} + S_{,\theta\theta}T_{,xx} \tag{13}$$

with $p = h, g, q$ given in terms of a_{ij}, b_{ij}, and d_{ij} as per Eqs. (5) in Ref. 15. The displacements u and v in the boundary conditions were eliminated by a procedure following Ref. 16, with the conditions on u and v replaced by conditions on their derivatives.

The advantage of the w-F formulation over its u-v-w counterpart lies mainly in reducing the number of unknown functions from three to two. On the other hand, it is restricted to Donnell-type equations, and the conditions on u and v cannot be fully satisfied, which can lead to a discrepency for the lowest buckling load in composite laminated cylindrical shells.

Buckling Equations

The buckling equations are straightforward and derived with the aid of the perturbation technique:

$$u = u^{(0)} + \xi u^{(1)}, \qquad v = v^{(0)} + \xi v^{(1)}$$

$$w = w^{(0)} + \xi w^{(1)}, \qquad F = F^{(0)} + \xi F^{(1)} \tag{14}$$

where ξ is the load parameter, as the normalized amplitude of the buckling mode. The superscripts $^{(0)}$ and $^{(1)}$ denote the prebuckling and buckling states, respectively. Applying Eq. (14), pertubation of the differential operators yields

$$L(S) = L\left[S^{(0)}\right] + \xi L\left[S^{(1)}\right]$$

$$LL(S,T) = LL\left[S^{(0)}, T^{(0)}\right] + \xi\left\{LL\left[S^{(0)}, T^{(1)}\right] + LL\left[S^{(1)}, T^{(0)}\right]\right\} + \xi^2(\cdots)$$

$$LLL(S,T,S) = LLL\left[S^{(0)}, T^{(0)}, S^{(0)}\right] + \xi\left\{LLL\left[S^{(0)}, T^{(0)}, S^{(1)}\right] + LLL\left[S^{(0)}, T^{(1)}, S^{(0)}\right] + LLL\left[S^{(1)}, T^{(0)}, S^{(0)}\right]\right\} + \xi^2(\cdots) + \cdots \tag{15}$$

Substitution of Eqs. (14) and (15) in Eqs. (9) and (12) yields the partial differential equations of the prebuckling and buckling states. The set of partial differential equations is first reduced to a set of ordinary ones by seperation of the variables, namely,

$$u(x,\theta) = \sum_{m=0}^{2Nu} u_m(x)g_m(\theta), \qquad v(x,\theta) = \sum_{m=0}^{2Nv} v_m(x)g_m(\theta)$$

$$w(x,\theta) = \sum_{m=0}^{2Nw} w_m(x)g_m(\theta), \qquad F(x,\theta) = \sum_{m=0}^{2NF} f_m(x)g_m(\theta) \tag{16}$$

where $2Nu$, $2Nv$, $2Nw$, and $2NF$ are the numbers of retained terms in the relevant truncated series, and

$$g_m(\theta) = \begin{cases} \cos \mathrm{im}\,\theta & m = 0, 1, 2, 3, \ldots, N_3 \\ \sin \mathrm{im}\,\theta & m = N_3 + 1, \ldots, 2N_3 \end{cases} \tag{17}$$

i denotes the characteristic circumferential wave number (see Ref. 13). $N_3 = N_u$ or N_v or NF according to the equation number. On application of the Galerkin procedure for minimizing the errors

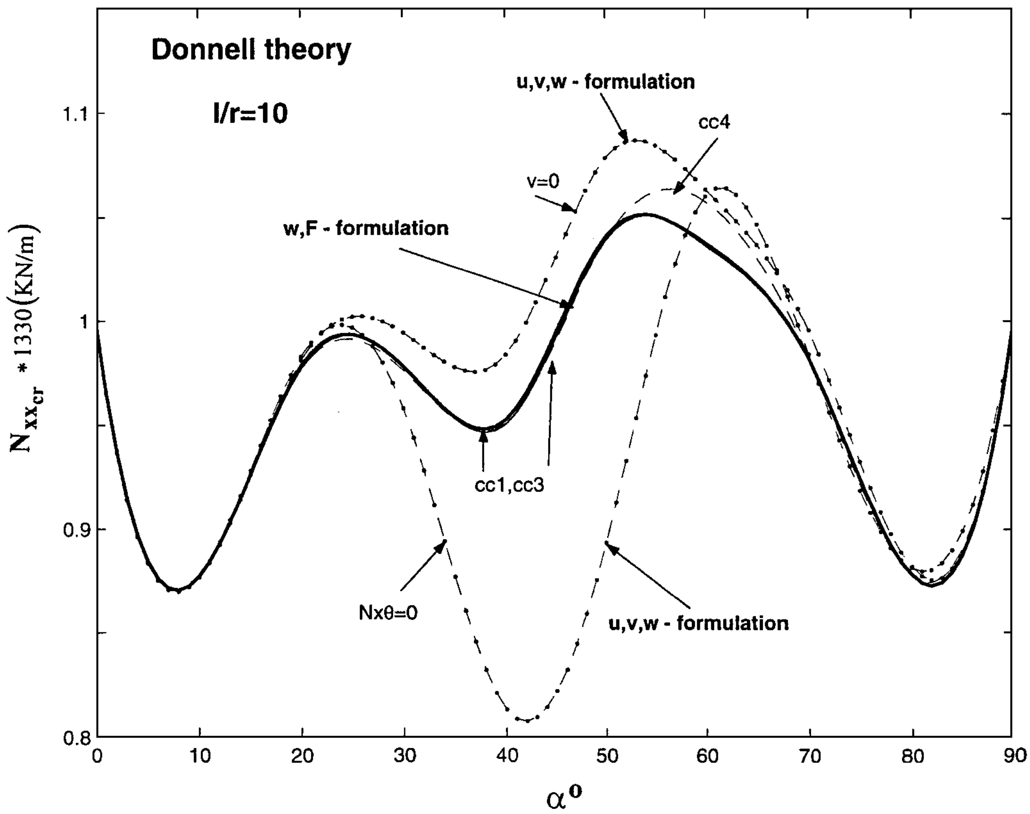

Fig. 3 Buckling load vs angle-ply ($\pm\alpha$) for $\ell/R = 10$ with clamped boundary conditions.

caused by the truncated form of the series, the following integrals must vanish:

For the u-v-w formulation

$$\int_0^{2\pi} [\text{equilibrium in } u, \text{Eq. (7a)}] g_p(\theta)\, d\theta = 0 \qquad p = 0, 1, \ldots, 2Nu$$

$$\int_0^{2\pi} [\text{equilibrium in } v, \text{Eq. (7b)}] g_p(\theta)\, d\theta = 0 \qquad p = 0, 1, \ldots, 2Nv$$

$$\int_0^{2\pi} [\text{equilibrium in } w, \text{Eq. (7c)}] g_p(\theta)\, d\theta = 0 \qquad p = 0, 1, \ldots, 2Nw \tag{18}$$

and for the w-F formulation

$$\int_0^{2\pi} [\text{compatibility, Eq. (12a)}] g_p(\theta)\, d\theta = 0 \qquad p = 0, 1, \ldots, 2NF$$

$$\int_0^{2\pi} [\text{equilibrium, Eq. (12b)}] g_p(\theta)\, d\theta = 0 \qquad p = 0, 1, \ldots, 2Nw \tag{19}$$

$g_p(\theta)$ are the weighting functions, chosen as $\cos(\text{im}\,\theta)$ and $\sin(\text{im}\,\theta)$. In the same way the Galerkin procedure is applied for the boundary condition as well.

The present theory assumes a linear prebuckling behavior, which yields an eigenproblem. Finally, a central finite difference scheme is used to reduce the ordinary differential equations to the following algebraic ones:

For the prebuckling state

$$[K]\{Z\} = \{P\} \tag{20}$$

and for the buckling state

$$\{[K] + \lambda[G]\}\{Z\} = 0 \tag{21}$$

where K and G are the stiffness and geometry matrices, respectively; Z is an unknown vector consisting of u, v, w, $u_{,xx}$, $v_{,xx}$, and $w_{,xx}$ for the u-v-w formulation and of w, f, $w_{,xx}$, $f_{,xx}$ for the w-F. Equation (21) is an eigenvalue problem in which λ represents the buckling load parameters and Z the buckling mode.

A general computer code was written, covering the prebuckling and buckling behavior of any laminated cylindrical shell and using both the u-v-w and w-F formulations.

Results and Discussion

The parametric study of the buckling load had two main purposes: 1) examination of the relative accuracies of the w-F and u-v-w formulations and 2) identification and analysis of the accuracy parameters with the aid of several shell theories. In addition, the effect of bending–stretching coupling was studied. For these purposes an angle-ply ($\pm\alpha$) graphite/epoxy cylindrical shell was taken from Ref. 13 with data as follows: 2-ply laminate with $E_{11} = 1.404.10^{11}$ N/m^2, $E_{22} = 0.973.10^{10}$ N/m^2, $G_{12} = 0.411.10^{10}$ N/m^2 ($E_{11}/E_{22} = 14.4$; $E_{11}/G_{12} = 34.1$), $\nu_{12} = 0.26$, radius $R = 1.27$ m, thickness $h = 0.0127$ m ($R/h = 100$). The modes of loading studied were axial compression and torsion.

Axial Compression

The axial compression load was applied through the boundary condition by setting $N_{xx} = \bar{N}_{xx}$ at one end. In the w-F formulation the boundary conditions are denoted SS_i and CC_i for the simply supported and clamped-clamped versions, respectively, ($i = 1$ for $N_{xx} = \bar{N}_{xx}$, $N_{x\theta} = 0$, $i = 2$ for $u = 0$, $N_{x\theta} = 0$, $i = 3$ for $N_{xx} = \bar{N}_{xx}$, $v = 0$ and $i = 4$ for $u = v = 0$). In the u-v-w formulation the boundary conditions are defined explicitly [see Eq. (8)].

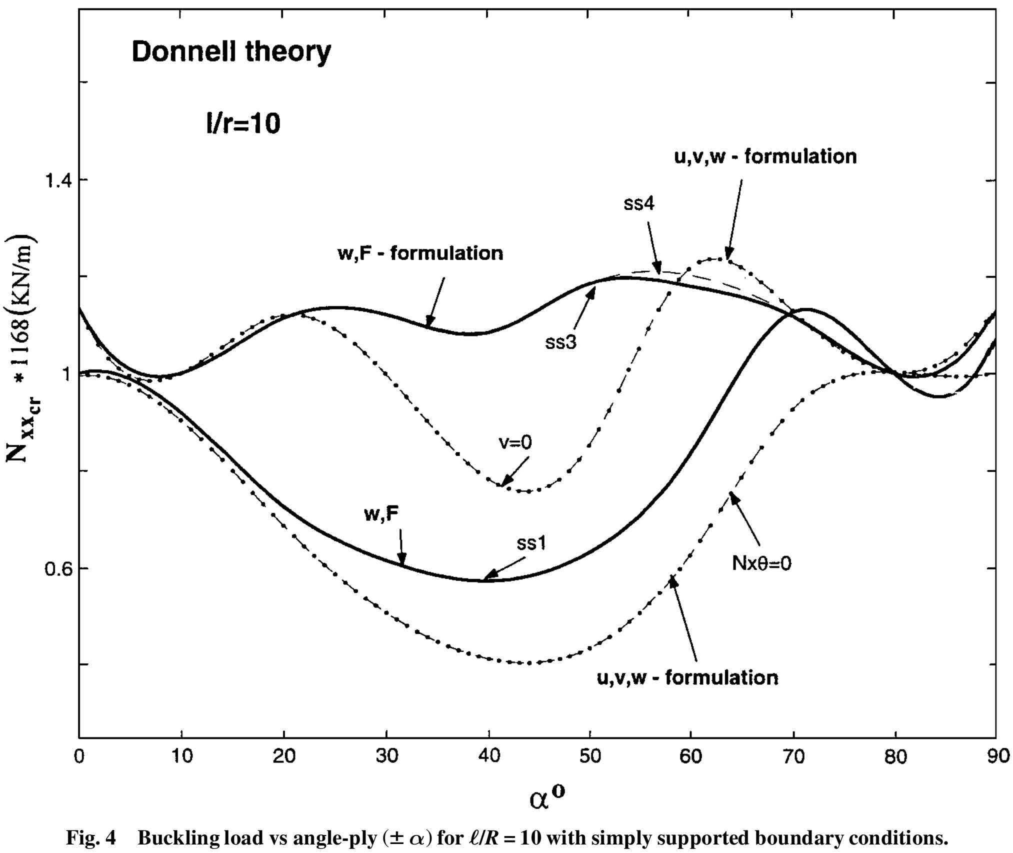

Fig. 4 Buckling load vs angle-ply ($\pm\alpha$) for $\ell/R = 10$ with simply supported boundary conditions.

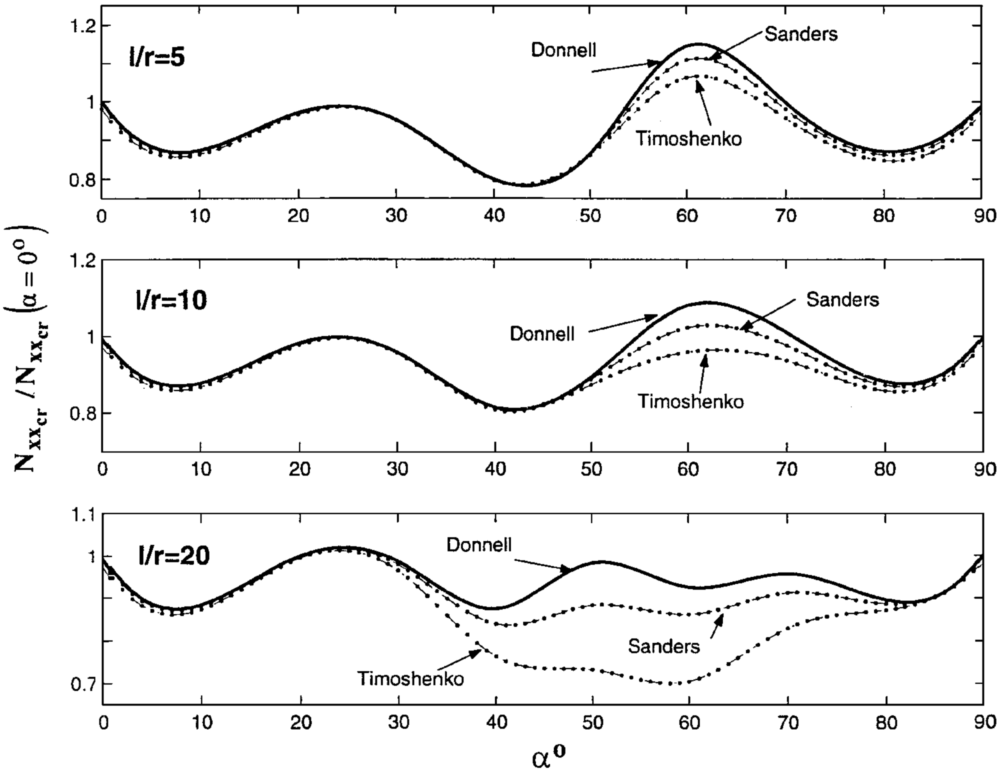

Fig. 5 Buckling load vs angle-ply (± α) for clamped conditions with various shell theory.

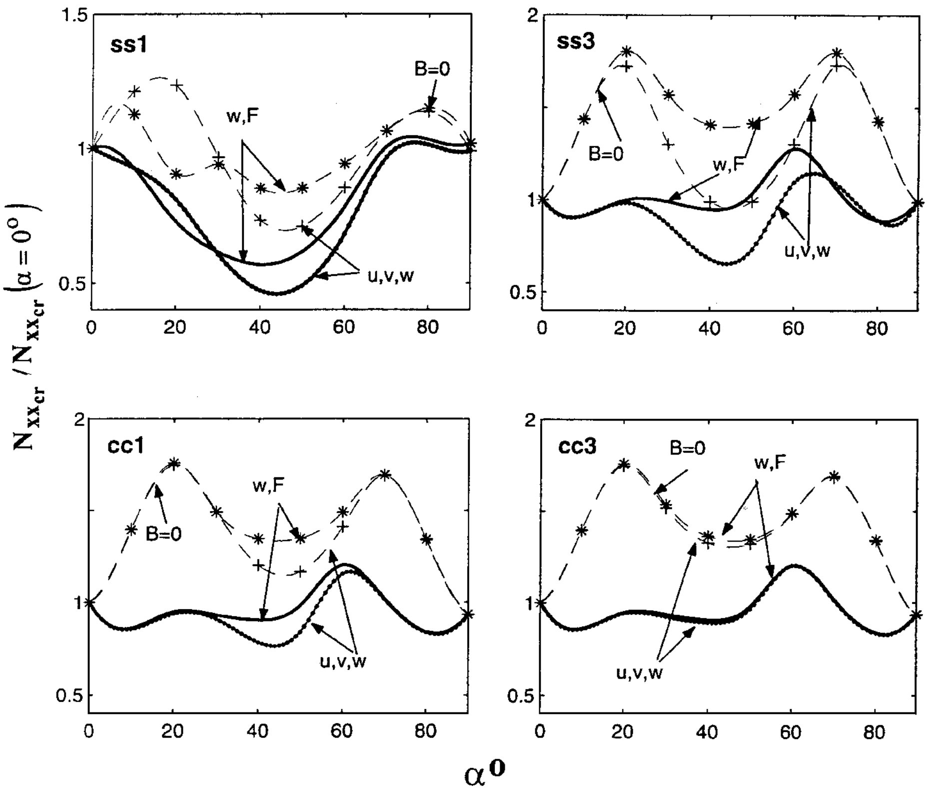

Fig. 6 Effect of stretching–bending coupling on buckling load for $\ell/R = 2$.

The buckling loads according to the two formulations are compared in Figs. 1 and 2 for $\ell/R = 2$ and in Figs. 3 and 4 for $\ell/R = 10$. For $\ell/R = 2$ the results, with the circumferential wave-number buckling mode, are also summarized in Table 1. They show a significant discrepancy between the formulations in the $30 \leq \alpha \leq 60$ range. For the CC versions the dominant discrepancy is pronounced under the in-plane condition of $N_{x\theta} = 0$ (CC_1) rather than under $v = 0(CC_3)$, whereas for the SS versions the picture is reversed. The reason for this situation was found to be the internal shear force caused by the laminated layup because of which the w-F formulation just missed the lowest eigenvalue. The second eigenvalue and eigenvector of the u-v-w formulation approach the first of the w-F. The w-F formulation is unsuitable for all arbitrary stacking combination and orientation.

In Fig. 5 the buckling load is plotted vs the angle-ply ($\pm\alpha$) for $\ell/R = 5$, 10, and 20 under the different shell theories. It is seen that

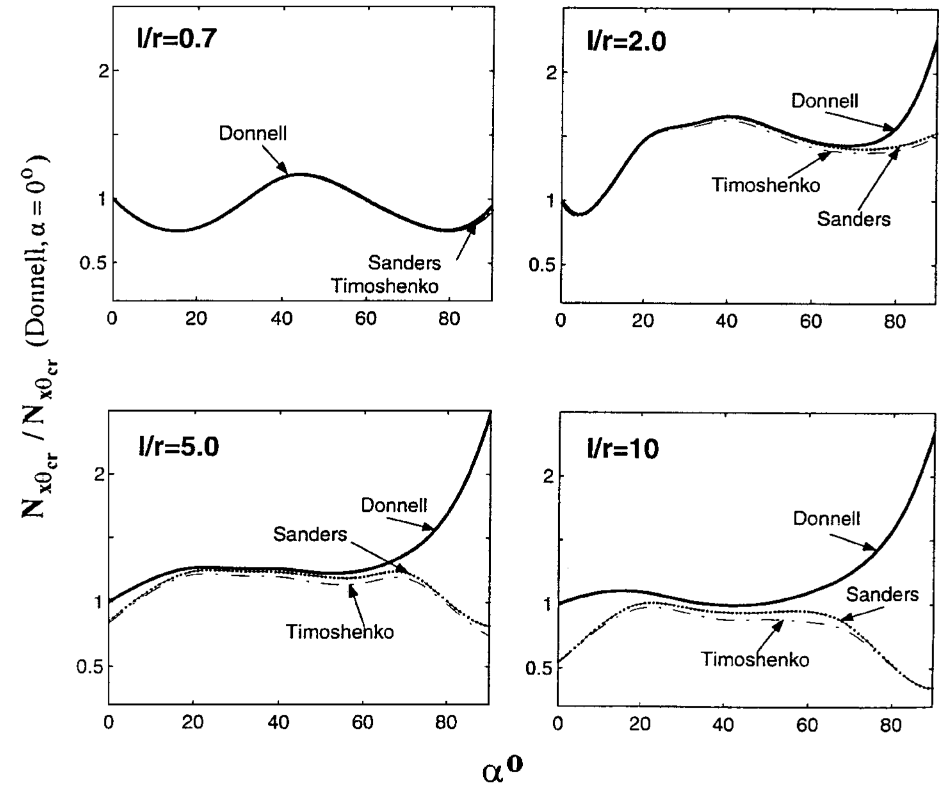

Fig. 7 Torsional buckling load vs angle-ply (± α) for clamped boundary conditions.

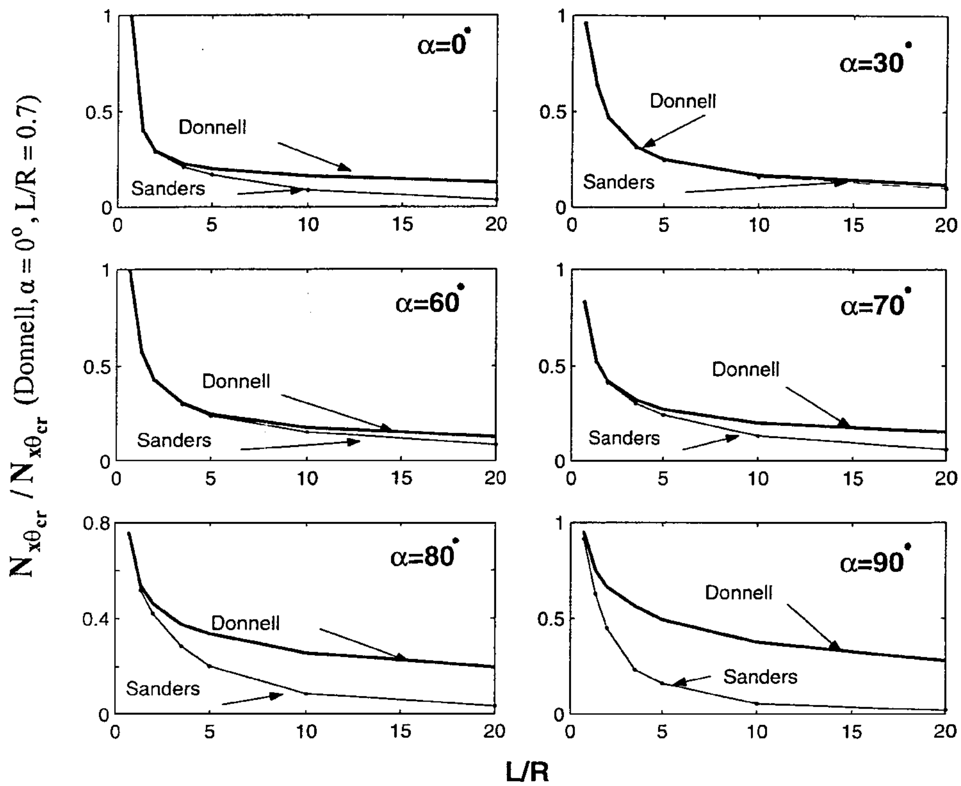

Fig. 8 Comparison of torsional buckling load obtained by Donnell and Sanders shell theories.

the choice of the most accurate theory depends on α and that the curves gradually diverge as ℓ/R increases. Again for all layups the simplified Donnell theory is very limited, the Sanders one is less so, and the most accurate theory is that of Timoshenko.

The stretching–bending coupling effect was examined in terms of the B_{ij} (B_{13}, B_{23}, B_{31}, B_{32}). By artificial zeroing of the B_{ij}, one can conclude the following about its effect. In Fig. 6 the buckling load for $B_{ij} = 0$ (Donnell theory) is plotted (dashed lines) vs the angle ply for different boundary condition with $\ell/R = 2$ and compared to the real case of $B_{ij} \neq 0$ (solid lines). It is seen that the buckling load is significantly reduced by the B_{ij} components. It is also seen that the characteristic discrepancy between the w-F and u-v-w formulations is retained in principle; hence, it is not a function of B_{ij}. The buckling loads for $\alpha = 0$ are $N_{xx_{cr}} = 1142(6)$, 1340 (8), 1420 (8), and 1425 (8) kN/m for $SS1$, $SS3$, $CC1$, and $CC3$, respectively.

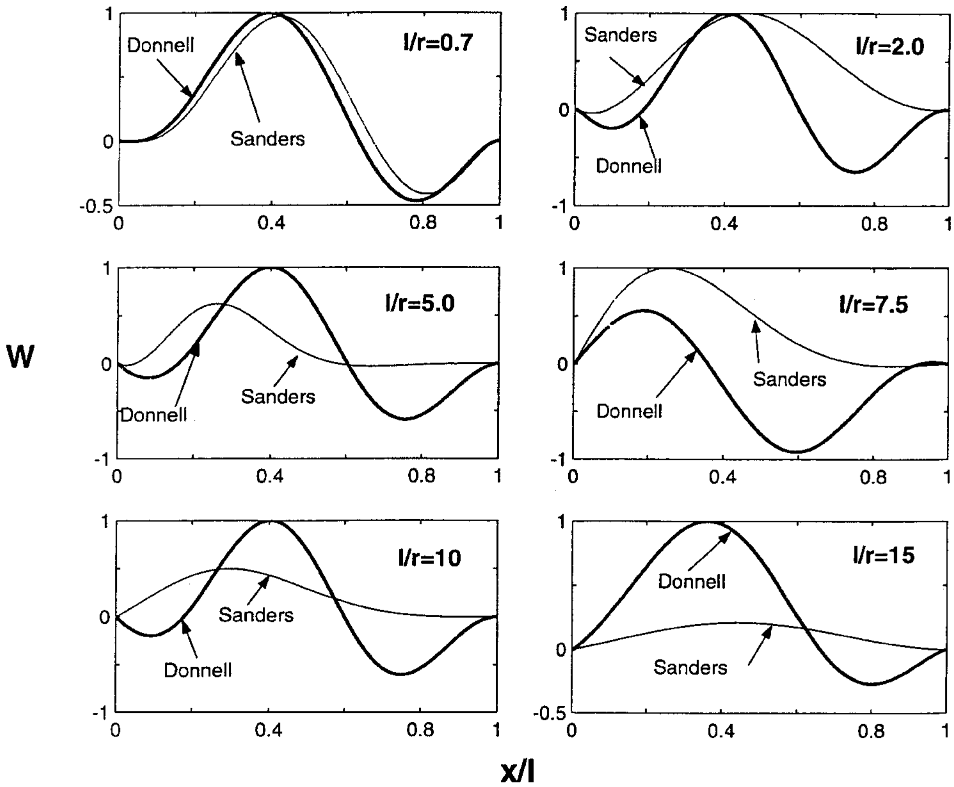

Fig. 9 Torsional buckling mode of transverse w-displacement for $\alpha = 90$ deg at $\theta = 0$ deg.

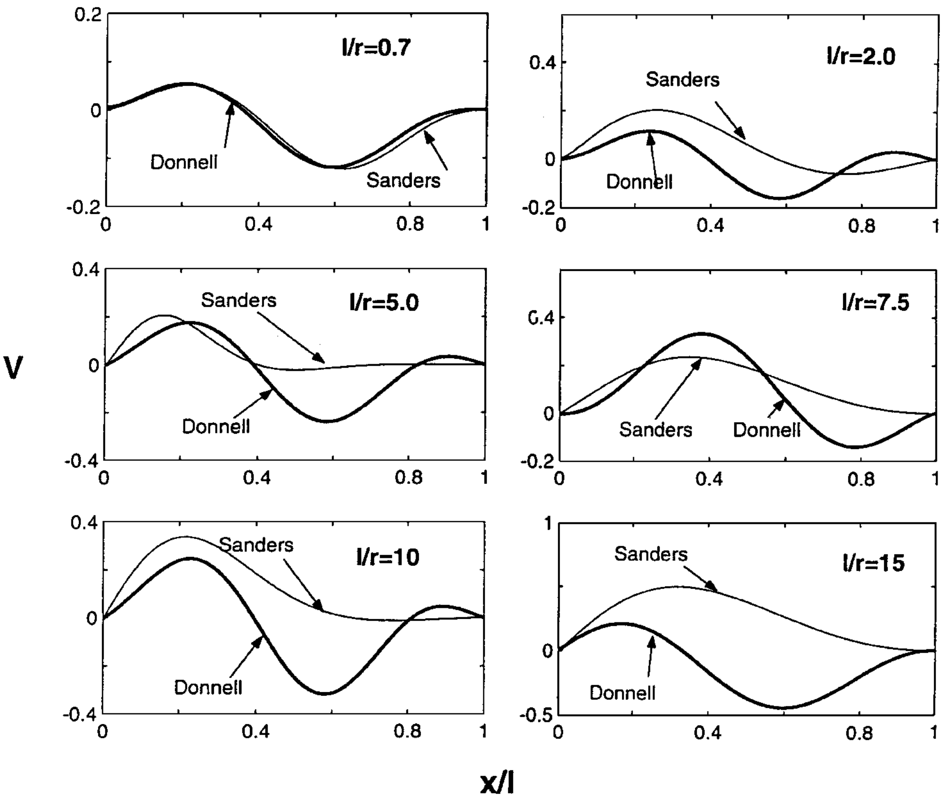

Fig. 10 Torsional buckling mode of inplane v displacement for $\alpha = 90$ deg at $\theta = 0$ deg.

Torsion

The torsional load was applied through the boundary condition by setting $w = w_{,x} = u = v = 0$ at one end and $w = w_{,x} = N_{xx} = 0$, $N_{x\theta} = \bar{N}_{x0}$ at the other. Only the u-v-w formulation was examined. The normalized buckling load is plotted vs the angle ply in Fig. 7 for several ℓ/R ratios, according to the three theories. It is seen that for small ℓ/R ratios (e.g., $\ell/R < 1$) the Donnell theory is quite accurate, for medium ratios ($1 \leq \ell/R \leq 5$) the Sanders theory is suitable over a certain interval of α, and for large ratios ($\ell/R > 5$) over the whole range of α. It should be realized that the buckling loads converge from above and the more accurate the theory, the lower the buckling load. The buckling load is plotted vs the length-to-radius ratio in Fig. 8 under the Donnell and Sanders theories for different α. It is seen that the discrepancy depends on the α, for example the Sanders theory is suitable at $\ell/R > 10$ for $\alpha = 0$ deg and at $\ell/R > 1$ for $\alpha = 90$ deg. For the latter layup the buckling

Table 1 Axial compression buckling load ($N_{xx_{cr}}$, kN/m) for $\ell/R = 2.0$

	w-*F* formulation						*u*, *v*, *w* formulation			
							$w = w_{,x} = 0$		$w = M_{xx} = 0$	
$\pm\alpha$	CC1	CC3	CC4	SS1	SS3	SS4	$N_{x\theta} = 0$	$v = 0$	$N_{x\theta} = 0$	$v = 0$
0	1425 (8)[a]	1426 (8)	1426 (8)	1142 (6)	1342 (8)	1345 (8)	1426 (8)	1427 (8)	1161 (7)	1345 (8)
10	1227 (8)	1229 (8)	1228 (8)	1057 (7)	1224 (8)	1224 (8)	1223 (8)	1228 (8)	1084 (6)	1230 (7)
20	1351 (6)	1352 (6)	1350 (6)	832 (1)	1340 (6)	1341 (6)	1341 (6)	1349 (6)	947 (5)	1324 (6)
30	1326 (3)	1341 (3)	1342 (4)	701 (1)	1330 (2)	1331 (2)	1301 (3)	1327 (3)	709 (4)	1170 (4)
40	1289 (3)	1299 (3)	1229 (3)	648 (1)	1273 (2)	1273 (2)	1121 (3)	1280 (3)	551 (3)	914 (4)
50	1399 (4)	1399 (4)	1400 (4)	703 (1)	1388 (2)	1388 (2)	1201 (3)	1387 (3)	574 (3)	986 (2)
60	1713 (3)	1717 (3)	1717 (3)	892 (1)	1714 (2)	1715 (2)	1650 (3)	1722 (3)	797 (3)	1417 (7)
70	1450 (8)	1458 (8)	1480 (8)	1150 (1)	1431 (8)	1479 (8)	1456 (8)	1458 (8)	1132 (3)	1467 (8)
80	1198 (8)	1198 (8)	1202 (8)	1179 (8)	1195 (8)	1200 (8)	1199 (8)	1200 (8)	1187 (8)	1198 (9)
90	1333 (8)	1333 (8)	1333 (8)	1165 (6)	1327 (8)	1328 (8)	1333 (8)	1333(8)	1161 (6)	1328 (8)

[a]Numbers in parentheses refer to the circumferential wave.

modes of w and v, along the axial direction at the circumferential coordinate $\theta = 0$, are plotted in Figs. 9 and 10, respectively. Except for the low length-to-radius ratios, significant differences are seen between the two theories even for layups yielding the almost the same buckling load.

Conclusions

A buckling analysis and a solution procedure are presented for laminated cylindrical shells for three different shell theories. For the Donnell-type equations a formulation based on Airy stress function was checked out compared to the more accurate u-v-w formulation. From the results presented the following conclusions can be drawn:

1) Choice of the most accurate shell theory depends on the geometrical and physical aspect ratios.

2) The stretching–bending coupling effect is very pronounced.

3) Unlike the isotropic shell, the formulation based on the Airy stress function may be inaccurate in terms of the buckling load.

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